The interpretation of powersets in the context of (0,1)-category theory and its applications to point-set topology

Suppose we wish to do topology in the setting of a category. There is—of course—a well-known solution to this is: put a Grothendieck topology on the category and work with the category of sheaves on the resulting site. This makes the objects in our category behave like the open subsets of a topological space, and this is why sites (or, more properly, Grothendieck toposes) are analogous to locales.

From the point of view of point-set topology it's easy to motivate focusing on the behavior of open sets and abandoning points all together: sets are unstructured collections, and a priori no pioint has a meaningfully distinct relationship to any other point in the same set until we impose some additional structure on said set—the obviously-relevant example here being a topology. Thus for most topological situations the points don't carry much information.

But now suppose that we want to put a topology on a category, but in such a way that the objects of our category behave like the *points* of a topological space rather than the open sets. It's no longer clear how to proceed, largely for two reasons.

1. We need some analog of the powerset as a place for open sets to live.
2. And, more importantly, this place needs to adequately reflect the arrows between our objects—something a set doesn't have, and consequently something a powerset doesn't need.

The latter point is suggestive that—beyond perhaps groupoids, as categorified sets—the naive approach of trying to work with a 2-category of all subcategories may not be the right analog of a powerset. We don't know for certain *how* the setting we're trying to pursue should reflects arrows. To do this we need more information, and a good place to look for this is (0,1)-categories, i.e. preordered sets. Not only is this the simplest case where non-trivial arrows exist between points, but it's also a case where relations to topology are well-known.

Note. Throughout, I will allow **Pos**—typically the category of partially-ordered sets with monotone maps as morphisms—to also include preordered sets. In my opinion this isn't a significant change, and **Pos** as notation is a bit easier to pick out at a glance than options for preordered sets. Focusing on preordered sets is also the correct choice since we will be focusing on the perspective of (0,1)-categories, where a preordered set and a partially-ordered sets are essentially the same thing.

1 Connections between preordered sets and point-set topology

1.1 The Alexandrov topology

Given a preordered set X the Alexandrov topology $\mathrm{Alx}(X)$ on X has as open subsets the upper subsets of X. This is essentially the canonical topology on a preordered set, and it has a number of nice properties. Let $F : \mathbf{Pos} \to \mathbf{Set}$ be the forgetful functor.

- Let $X, Y \in \mathbf{Pos}$ be arbitrary. Then a function $F(X) \to F(Y)$ between the underlying subsets gives a continuous map $\mathrm{Alx}(X) \to \mathrm{Alx}(Y)$ if and only if that function gives a monotone map $X \to Y$. Thus we have not only a functor $\mathrm{Alx} : \mathbf{Pos} \to \mathbf{Top}$, but this functor is even fully-faithful.
- Moreover, Alx arises canonically at the left adjoint of a particular functor $\mathbf{Top} \to \mathbf{Pos}$.
- Each of the collection of upper and lower subsets of a preordered set form a complete lattice (possibly only up to a suitable notion of equivalence), much like powersets. Nevertheless there are important differences, e.g. the collection of lower/upper subsets is not generally atomistic.
- While the collections of upper and lower subsets are not generally self-dual, unlike a powerset, they are duals of one another.
- The interior of an arbitrary subset $A \subseteq X \in \mathbf{Pos}$ is recovered as the corresponding upper subset $A \uparrow \subseteq X$.
- Dually, the closure of $A \subseteq X \in \mathbf{Pos}$ is given by the lower subset $A \downarrow \subseteq X$.

Notably, the first fact listed is implied by the following observation: if $f : X \to Y$ is a monotone map and $U \subseteq Y$ is an upper subset then $f^{-1}(U) \subseteq X$ is also an upper subset. This is of course similar to the defining property of a continuous map. Likewise, the preimage of a lower subset is again lower.

Certain canonical constructions assigning topologies to sets may be “carried through” **Pos** in a sense, because the resulting topology is a special case of the Alexandrov topology. More specifically, let $\mathrm{Fr}, \mathrm{Cofr} : \mathbf{Set} \to \mathbf{Pos}$ be the free and cofree functors, that is the left and right adjoints to the forgetful functor F. These constructions are straightforward, and amenable to explicit description. In both of the following let $A \in \mathbf{Set}$ be arbitrary.

- The free poset $\mathrm{Fr}(A)$ on A is the poset (specifically, rather than a preordered set) such that for all $x, y \in A$, $x \leq y$ if and only if $y \leq x$. More explicitly, the order relation on $\mathrm{Fr}(A)$ is equality.
- The cofree preorder $\mathrm{Cofr}(A)$ on A is the preordered set such that for all $x, y \in A$, $x \leq y$.

Let us consider the Alexandrov topologies $\mathrm{Alx}(\mathrm{Fr}(A))$ and $\mathrm{Alx}(\mathrm{Cofr}(A))$ of $\mathrm{Fr}(A)$ and $\mathrm{Cofr}(A)$, respectively. In $\mathrm{Fr}(A)$ no two distinct elements of A are comparable, ergo any subset of A an upper subset of $\mathrm{Fr}(A)$. Consequently, the open subsets of $\mathrm{Alx}(\mathrm{Fr}(A))$ are every subset of A, i.e. we have the discrete topology. On the other hand, any non-empty upper subset of $\mathrm{Cofr}(A)$ must include every point of A, so the only open subsets of $\mathrm{Alx}(\mathrm{Cofr}(A))$ are $\varnothing$ and A itself. This is, of course, the indiscrete topology on A.

The relevance of this to the overall discussion is that the discrete and indiscrete topology functors Disc, Indsc : **Set** $\to$ **Top** also arise as the left and right adjoints of the forgetful functor **Top** $\to$ **Set**. These are compatible with the Alexandrov topology in the sense that we have natural isomorphisms $\mathrm{Alx} \circ \mathrm{Fr} \xrightarrow{\sim} \mathrm{Disc}$ and $\mathrm{Alx} \circ \mathrm{Cofr} \xrightarrow{\sim} \mathrm{Indsc}$.

1.2 The specialization preorder

Let (X, T) be a topological space. We can give X the structure of a preordered set $(X, \leq)$, called the *specialization preorder* by declaring for all $u, v \in X$ that $u \leq v \iff u \in \overline{\{v\}}$. Here $\overline{\{v\}}$ denotes the closure of the singleton $\{v\}$. If $f : X \to Y$ is a continuous map then f passes to a monotone map between the resulting preorders. We thus have a faithful functor $\mathrm{Sp} : \mathbf{Top} \to \mathbf{Pos}$ where open subsets of X correspond to upper subsets of $\mathrm{Sp}(X)$ and closed subsets of X correspond to lower subsets is $\mathrm{Sp}(X)$.

Importantly, Alx is left adjoint to Sp. The composition $G = \mathrm{Alx} \circ \mathrm{Sp} : \mathbf{Top} \to \mathbf{Top}$ gives a comonad with the property that for every space X the space $G(X)$ is the specialization preorder with the Alexandrov topology applied to it. The counit $\eta : G \implies \mathrm{Id}_{\mathbf{Top}}$ is moreover a continuous bijection at every component $\eta_X : G(X) \to X$. As a consequence, we have the following significant result:

Theorem 1.1. *Every topological space is a coarsening of some Alexandrov topology, where the underlying preorder of the topology is the specialization preorder of the original space.*

Ergo, collections of upper subsets (and dually, of course, closed subsets) naturally underlie any topological space.

1.3 The Sierpinski space S and the ordinal 2

Recall that the Sierpinski space $\mathbf{S}$ is the two point space $\{0, 1\}$ with opens $\{\{0, 1\}, \{1\}, \varnothing\}$. This space is important because it classifies open subsets of arbitrary topological spaces.

More precisely, for any open subset $U \subseteq X$ of a topological space X there is a unique continuous map $f : X \to \mathbf{S}$ such that $f^{-1}(\{1\}) = U$, which occurs because $\{1\}$ is the unique open subset of $\mathbf{S}$ which is not closed. And of course a dual statement applies as well: given any closed subset $V \subseteq X$ there is a unique $g : X \to \mathbf{S}$ such that $g^{-1}(\{0\}) = V$. As before, this occurs because $\{0\}$ is the unique closed subset of $\mathbf{S}$ which is not open. These two facts are related by an obvious duality: if $X \setminus U = V$ then $g = f$.

The ordinal $\mathbf{2}$, sometimes also called the poset of truth values depending on context, is the unique (up to isomorphism) 2-element total order. That is, e.g., $\mathbf{2} = \{0, 1\}$ with the order relations $0 \leq_{\mathbf{2}} 0$, $0 \leq_{\mathbf{2}} 1$, and $1 \leq_{\mathbf{2}} 1$. Similarly to $\mathbf{S}$, a useful property of $\mathbf{2}$ is that it classifies upper subsets of preordered sets: if $U \subseteq X$ is an upper subset of the preordered set X then there is a unique monotone map $f : X \to \mathbf{2}$ such that $f^{-1}(\{1\}) = U$. And dually, for every lower subset $L \subseteq X$ there is a unique monotone map $g : X \to \mathbf{2}$ such that $g^{-1}(\{0\}) = L$. If $X \setminus U = L$ then $f = g$.

These analogies are expected, because $\mathrm{Sp}(\mathbf{S}) \simeq \mathbf{2}$ and $\mathrm{Alx}(\mathbf{2}) \simeq \mathbf{S}$—that is, they are adjointly equivalent objects. I would argue, going further than this, that the two objects are measuring the same things.

2 What conclusions to draw?

Given some $X \in \mathbf{Pos}$ let $\Delta(X)$ be its preordered set of upper subsets and $\nabla(X)$ its preordered set of lower subsets. We will respectively refer to these as the *upper Alexandrov preorder* and *lower Alexandrov preorder* of X.

Given what has been presented thus far, how should we interpret $\Delta(X)$ and $\nabla(X)$ in the context of a categorical perspective on topological spaces? I propose that the simplest way of understanding them is that they are—taken together as a pair—generalizations of powersets for the case where the our points have arrows between them. Each encode how our points related to one another as imposed by the arrows, and they do this in opposite but related ways.

- The upper Alexandrov preorder of a given preordered set consist of the *open-admissible subsets*.
- In turn, the lower Alexandrov preorder consists of the *closed-admissible suborders*.

That is to say, upper subsets of a preordered set are the subobjects of that preordered set that can be admitted as open subsets in some topological space. And of course the dual statement holds as well: lower subsets are those subobjects that can be admitted as closed subsets in some topological space. When our preordered set X has no non-trivial relations between points—i.e., when it is a free poset—then the result that $\Delta(X) \simeq \nabla(X) \simeq 2^X$ can be seen to reflect that any subobject may be admitted as either open or closed in some topology.

Under this interpretation, the Alexandrov topology is consequently understood as the simplest choice among all possible topologies on a preordered set: we accept as open subsets *all* possible open subsets of our preordered set; or, dually, we accept as closed subsets all possible closed subsets of our preordered sets. This motivates, for example, why a free poset should generate the discrete topology, why a cofree preorder should generate the indiscrete topology, or why the Alexandrov topology functor is left adjoint to the specialization preorder functor.

2.1 Alexandrov preorders, functors, and monads

Let $f : X \to Y \in \mathbf{Pos}$ be some arbitrary monotone maps. This induces a pair of canonical morphisms between the preorders $\Delta(X)$ and $\Delta(Y)$, and dually also a pair of morphisms between $\nabla(X)$ and $\nabla(Y)$. For simplicitly let us consider only the upward case. The most obvious manner of turning f into a morphism between the upper Alexandrov preorders is in the backwards direction:

$$\begin{aligned} f^* : \Delta(Y) &\to \Delta(X) \\ U &\mapsto \operatorname{preim}_f(U) \end{aligned}$$

which follows from a previously-stated fact, which is that the preimage of an upper subset with respect to a monotone map is again an upper subset. Less obvious is the forwards direction:

$$\begin{aligned} f_* : \Delta(X) &\to \Delta(Y) \\ U &\mapsto \operatorname{im}_f(U){\uparrow} \end{aligned}$$

This works because every subset of a preordered set has a unique closure as an upper subset.

These two induce a pair of functors $\Delta_* : \mathbf{Pos} \to \mathbf{Pos}$ and $\Delta^* : \mathbf{Pos}^{\mathrm{op}} \to \mathbf{Pos}$, and the obvious analogous cases induce functors $\nabla_* : \mathbf{Pos} \to \mathbf{Pos}$ and $\nabla^* : \mathbf{Pos}^{\mathrm{op}} \to \mathbf{Pos}$. These reflect facts about the powerset functor 2^S for $S \in \mathbf{Set}$. And furthermore, much like the powerset functor, we have that the covariant functors are in fact monads such that the components of the units of the monads

$$\eta_X^{\Delta} : X \hookrightarrow \Delta_*(X)$$
$$\eta_X^{\nabla} : X \hookrightarrow \nabla_*(X)$$

are order embeddings, where an element $x \in X$ is sent to the principal upset $\{x\}\uparrow$ and principal downset $\{x\}\downarrow$ respectively. There is more to be said on this topic—specifically how to interpret the upper subsets not picked out by elements of X—but this is beyond the particulars of this article.

3 Implications for higher categories

By "higher categories" we mean higher than (0,1)-categories. In this particular case we will only focus on 1-categories, and specifically those that are locally-small or (essentially) small.

We can interpret a (0,1)-category as a **2**-enriched category, so that for a given preordered set X we think of the hom of X as a functor $\mathrm{Hom}_X : X^{\mathrm{op}} \times X \to \mathbf{2}$. Given elements $u, v \in X$ the hom functor of X encodes the order relation: $\mathrm{Hom}_X(u, v) \iff u \leq_X v$. This gives us a way to identify principal upsets with the covariant hom functor and principal downsets with the contravaraint hom functor: $\mathrm{Hom}_X(-, v)$ corresponds to $\{v\}\downarrow$ and $\mathrm{Hom}_X(u, -)$ corresponds to $\{u\}\uparrow$. More generally the **2**-valued presheaves on X give the lower subsets of X and the **2**-valued copresheaves of X give the upper subsets.

In more general terms, this means $\Delta(X) = \mathrm{PSh}_{\mathbf{2}}(X)$ and $\nabla(X) = \mathrm{CoPSh}_{\mathbf{2}}(X)$, and the units η_X^{Δ} and η_X^{∇} are respectively the covariant Yoneda embedding $u \mapsto \mathrm{Hom}_X(u, -)$ and the contravariant Yoneda embedding $v \mapsto \mathrm{Hom}_X(-, v)$. This is suggestive of a straightforward way to develop upper and lower Alexandrov categories, i.e. functors $\Delta_*, \nabla_* : \mathbf{Cat} \to \mathbf{Cat}$, by simply using the presheaf and copresheaf categories. The problem is that **Set** is a large category, unlike **2**, which means that the Yoneda embedding is not an endofunctor on **Cat** and in turn non-monadic.

This gives us a start for developing Alexandrov categories in a higher-categorical setting and therefore a start for a more-direct analog of point-set topology in such a setting, but it requires care. It's not as immediate as in the (0,1)-category setting. There is more to say on this topic, but that is in turn a topic for a more in-depth discussion beyond the scope of this article.